EPFL– Fall 2025
Series 12

Differential Geometry I: Curves and surfaces

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A. Standard exercises

12.1 Prove that if $\gamma : I \to S \subset \mathbb{R}^3$ is a C^2 biregular curve that is a geodesic of a co-oriented surface S, then the torsion of γ (as a curve in $\mathbb{R}^3$) coincides up to sign with the *geodesic torsion* of this curve. (This result explains the terminology "geodesic torsion".) Is the converse statement true? (i.e. is a curve on a surface whose geodesic torsion equals its torsion always a geodesic?)

Hint. It is useful to compare the Darboux equations with the Serret–Frenet equations for the curve γ.

Solution. Let n denote the co-orientation of the surface, and let us denote for simplicity the restriction of n to the curve γ by $n(t) = n(\gamma(t))$. Write the Darboux equations and the Serret–Frenet equations for the curve γ:

$$\begin{cases} \frac{1}{V}\dot{T} = k_g\mu + k_n n, \\ \frac{1}{V}\dot{\mu} = -k_g T - \tau_g n, \\ \frac{1}{V}\dot{n} = -k_n T + \tau_g \mu, \end{cases} \qquad \begin{cases} \frac{1}{V}\dot{T} = \kappa N, \\ \frac{1}{V}\dot{N} = -\kappa T + \tau B, \\ \frac{1}{V}\dot{B} = -\tau N. \end{cases}$$

If γ is a geodesic, then its geodesic curvature k_g is zero. Then the first equation of the first triad becomes

$$\frac{1}{V}\dot{T} = k_n n$$

which, when combined with the first equation in the second triad yeilds

$$k_n n = \kappa N.$$

Since both n and N are unit vectors, we infer that

$$n = \pm N$$

as well as $k_n = \pm\kappa$. Then the third equation from the first triads becomes

$$\frac{1}{V}\pm\dot{N} = -k_n T + \tau_g \mu.$$

Combining it with the second equation from the second triad, we get that

$$\tau B = \pm\tau_g \mu,$$

from which we infer that $\mu = \pm B$ and $\tau = \pm\tau_g$.

The converse is false. For example, if S is a plane, then the torsion of any biregular curve γ contained in S is zero. The normal vector n to S is constant, so the geodesic torsion of γ is also zero (since $\tau_g(t) = \langle \dot{n}, \mu \rangle = 0$). The two torsions coincide (they are both zero), yet γ is not a geodesic (in this case γ is never a geodesic unless it is a straight line).

12.2 Consider the circular right cylinder

$$C \subset \mathbb{R}^3, \qquad x^2 + y^2 = a^2.$$

Compute its Gauss and mean curvature (say with respect to the co-orientation corresponding to the inwards pointing normal). Can you predict these values without performing calculations?

Solution. The Gauss curvature is the product of the principal curvatures, and the mean curvature is half their sum. We also know that the principal directions (i.e. directions of the principal curvatures) are orthogonal.

It is intuitively clear that all normal curvatures of a right circular cylinder have the same sign (negative or positive depending on the choice of co-orientation). Through each point of the cylinder passes a straight line, so the minimal (or maximal) curvature is zero. The other principal curvature is that of the circle of radius a. This curvature is equal to $\pm 1/a$ (sign depending on orientation). Therefore

$$K = 0, \qquad H = \pm\frac{1}{2a}.$$

Let us try to stablish the above rigorously. A parametrization of the cylinder is

$$\psi(u, v) = (a\cos u,\, a\sin u,\, v), \qquad (u, v) \in [0, 2\pi] \times \mathbb{R}.$$

The associated tangent basis is

$$b_1 = \frac{\partial \psi}{\partial u} = (-a\sin u, a\cos u, 0), \qquad b_2 = \frac{\partial \psi}{\partial v} = (0, 0, 1),$$

and the normal is

$$n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|} = (\cos u, \sin u, 0).$$

We observe that b_1 and b_2 are eigenvectors of the Weingarten map $L = dn$. Indeed

$$L(b_1) = \frac{\partial n}{\partial u} = (-\sin u, \cos u, 0) = \frac{1}{a} b_1,$$

$$L(b_2) = \frac{\partial n}{\partial v} = 0 = 0 \cdot b_2.$$

Thus, in the basis $\{b_1, b_2\}$,

$$L = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$K = \det(L) = 0, \qquad H = -\frac{1}{2}\,\mathrm{Trace}(L) = -\frac{1}{2a}.$$

If we choose the opposite co-orientation, we obtain $H = 1/(2a)$.

12.3 Consider the helicoid surface, which is parametrized by

$$\psi(u,v) = (v\cos u,\ v\sin u,\ u).$$

Compute the metric tensor, the second fundamental form, and the Weingarten map. What are the mean curvature and Gauss curvature of this surface?

Solution. For each $(u,v) \in \mathbb{R}^2$, a basis of $T_{\psi(u,v)}S$ is given by

$$b_1 = \frac{\partial\psi}{\partial u} = (-v\sin u, v\cos u, 1), \qquad b_2 = \frac{\partial\psi}{\partial v} = (\cos u, \sin u, 0),$$

and a unit normal is

$$n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|} = \frac{(-\sin u, \cos u, -v)}{\sqrt{1+v^2}}.$$

The metric tensor is

$$G(u,v) = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We compute

$$\frac{\partial^2\psi}{\partial u^2} = -v(\cos u, \sin u, 0), \qquad \frac{\partial^2\psi}{\partial u \partial v} = (-\sin u, \cos u, 0), \qquad \frac{\partial^2\psi}{\partial v^2} = (0,0,0).$$

Thus, using the formula $h_{ij} = \langle n, \partial^2_{ij}\psi\rangle$, we have

$$h_{11} = 0, \qquad h_{22} = 0, \qquad h_{12} = h_{21} = \frac{1}{\sqrt{1+v^2}}.$$

In matrix form:

$$H = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+v^2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}.$$

The Weingarten map is

$$L = -G^{-1}H = \begin{pmatrix} 0 & -(1+v^2)^{-3/2} \\ -(1+v^2)^{-1/2} & 0 \end{pmatrix}.$$

The Gauss curvature is

$$K = \det(L) = \frac{\det(H)}{\det(G)} = -\frac{1}{(1+v^2)^2}.$$

The mean curvature is

$$H = -\tfrac{1}{2}\,\mathrm{Trace}(L) = 0.$$

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12.4 Let S be the graph (in $\mathbb{R}^3$) of the C^2 function

$$\varphi : \Omega \to \mathbb{R}, \qquad \Omega \subset \mathbb{R}^2.$$

Show that the matrix of the second fundamental form is

$$H = \frac{1}{\sqrt{1+\varphi_x^2+\varphi_y^2}} \begin{pmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{pmatrix}.$$

Solution. The graph of φ is the surface $z = \varphi(x, y)$, parametrized by

$$\psi(x, y) = (x, y, \varphi(x, y)).$$

We use the notation

$$\varphi_x = \frac{\partial \varphi}{\partial x}, \quad \varphi_y = \frac{\partial \varphi}{\partial y}, \quad \varphi_{xx} = \frac{\partial^2 \varphi}{\partial x^2}, \quad \varphi_{yy} = \frac{\partial^2 \varphi}{\partial y^2}, \quad \varphi_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y}.$$

The associated tangent basis is

$$b_1 = (1, 0, \varphi_x), \qquad b_2 = (0, 1, \varphi_y),$$

and the unit normal is

$$n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|} = \frac{(-\varphi_x, -\varphi_y, 1)}{\sqrt{1+\varphi_x^2+\varphi_y^2}}.$$

Recall that the matrix of the second fundamental form has entries h_{ij}, where

$$h_{ij} \doteq h(b_i, b_j) = \langle n, \partial^2_{ij} \psi \rangle.$$

We compute

$$\frac{\partial^2 \psi}{\partial x^2} = (0, 0, \varphi_{xx}), \quad \frac{\partial^2 \psi}{\partial x \partial y} = (0, 0, \varphi_{xy}), \quad \frac{\partial^2 \psi}{\partial y^2} = (0, 0, \varphi_{yy}).$$

Thus

$$h_{11} = \frac{\varphi_{xx}}{\sqrt{1+\varphi_x^2+\varphi_y^2}}, \qquad h_{12} = \frac{\varphi_{xy}}{\sqrt{1+\varphi_x^2+\varphi_y^2}}, \qquad h_{22} = \frac{\varphi_{yy}}{\sqrt{1+\varphi_x^2+\varphi_y^2}}.$$

12.4 Let $S \subset \mathbb{R}^3$ be a C^2 surface and $p \in S$. Let $L_p : T_pS \to T_pS$ be the shape operator at p. As we have seen in class, L_p is self-adjoint (recall the definition of this notion). Is the matrix of L_p (associated to a local parametrization $\psi : \Omega \to S$ of S around p) always symmetric?

Solution. Let V be a finite dimensional vector space with an inner product. A linear map $L : V \to V$ is called *self-adjoint* if, for every $x, y \in V$, we have

$$\langle L(x), y\rangle = \langle x, L(y)\rangle.$$

Let $\{e_1, \dots, e_n\}$ e a basis of V. Let us denote by G the *Gram matrix* of this basis, namely the basis with elements $G_{ij} = \langle e_i, e_j\rangle$. Note that, if we express any vector $x \in V$ as a column matrix with elements the components of x in this basis, i.e.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad iff \qquad x = x_1 e_1 + \dots x_n e_n,$$

then the Gram matrix is the matrix expression of the inner product in this basis: We have for any $x, y \in V$:

$$\langle x, y\rangle = x^T G y$$

(it is easy to verify this: expand bilinearly the expression $\langle x, y\rangle = \langle \sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\rangle$.

With the above notation, if L is also used to denote the matrix of the linear map L in this basis (i.e. the matrix with elements $L_{ij} = (L(e_j))_i$), then the condition that L is self adjoint becomes as follows: For any $x, y \in V$,

$$(Lx)^T G y = x^T G L Y \quad \Leftrightarrow \quad x^T L^T G y = x^T G L y$$

and, therefore

$$L^T G = G L.$$

Since G is symmetric, the above is equivalent to the statement that the matrix $G \cdot L$ is symmetric

Note that, $\{e_i\}_{i=1}^n$ is an orthonormal basis if and only if $G = I$. In that case, the above condition means that L is self adjoint if and only if the corresponding matrix is symmetric. However, the above equivalence between self adjointness and symmetricity of the corresponding matrix breaks down once we consider bases which are not orthonormal.

In case of the parametrization of a surface S, the basis $\{b_1, b_2\}$ of $T_{\psi(u)}S$ so the matrix of the shape operator in this basis is not necessarily symmetric (see the case of the helicoid).

12.6 In this exercise, we are going to adopt the notations of Ex. 9.5 regarding ruled surfaces: Let $S \subset \mathbb{R}^3$ be a C^2 ruled surface parametrized by $\psi : I \times (-\epsilon, \epsilon) \to S$,

$$\psi(u, v) = \gamma(u) + b(u)v,$$

where $\gamma : I\mathbb{R}^3$ is a C^2 regular curve and $b : I \to \mathbb{R}^3$ is a non-zero C^2 vector field along γ.

- **(a)** Show that the Gauss curvature of S satisfies $K \leqslant 0$.
- **(b)** Show that $K = 0$ when $\dot\gamma, b, \dot b$ are linearly dependent. Verify that this is indeed the case when S is a conical surface (see Ex. 11.2).

Remark. Note that the surfaces in Ex. 12.2 and 12.3 are ruled surfaces and, hence, the above result applies to them.

Solution. (a) For any $(u,v)\in I\times(-\epsilon,\epsilon)$, a basis of the tangent space of S at the point $p=\psi(u,v)$ is given by

$$b_1=\frac{\partial\psi}{\partial u}=\dot\gamma(u)+v\dot b(u),$$
$$b_2=\frac{\partial\psi}{\partial v}=b(u),$$

with unit normal

$$n=\frac{b_1\times b_2}{\|b_1\times b_2\}}=\frac{\dot\gamma(u)\times b(u)+v\dot b(u)\times b(u)}{\|b_1\times b_2\|}.$$

Therefore, we can compute the second fundamental form using the fact that

$$h_{ij}=h(b_i,b_j)=\langle n,\partial^2_{ij}\psi\rangle.$$

Since

$$\partial^2_{uu}\psi=\ddot\gamma(u)+v\ddot b(u),\quad \partial^2_{uv}\psi=\dot b(u),\quad \partial^2_{vv}\psi=0,$$

we have

$$\begin{aligned}
h_{uu}&=\langle n,\partial^2_{uu}\psi\rangle=\frac{1}{\|b_1\times b_2\|}\langle\dot\gamma(u)\times b(u)+v\dot b(u)\times b(u),\ddot\gamma(u)+v\ddot b(u)\rangle,\\
h_{uv}&=\langle n,\partial^2_{uv}\psi\rangle=\frac{1}{\|b_1\times b_2\|}\langle\dot\gamma(u)\times b(u)+v\dot b(u)\times b(u),\dot b(u)\rangle=\frac{1}{\|b_1\times b_2\|}\langle\dot\gamma(u)\times b(u),\dot b(u)\rangle\\
h_{vv}&=\langle n,\partial^2_{vv}\psi\rangle=0
\end{aligned}$$

so the matrix of the second fundamental form takes the form

$$H)=\begin{bmatrix} h_{uu} & \frac{1}{\|b_1\times b_2\|}\langle\dot\gamma(u)\times b(u),\dot b(u)\rangle\\ \frac{1}{\|b_1\times b_2\|}\langle\dot\gamma(u)\times b(u),\dot b(u)\rangle & 0\end{bmatrix}.$$

In particular,

$$\det H=-\frac{1}{\|b_1\times b_2\|^2}\langle\dot\gamma(u)\times b(u),\dot b(u)\rangle^2\leqslant 0.$$

If L is the matrix of the shape operator (with respect to the basis b_1,b_2) and G the matrix of the metric tensor, then we have

$$L=-G^{-1}H,$$

so that the Gauss curvature satisfies

$$K\doteq\det(L)=\frac{\det(H)}{\det(G)}.$$

Note that the matrix G is always positive definite (this is always true for the Gram matrix of any basis); in particular, we have $\det(G) = \langle b_1, b_1\rangle\langle b_2, b_2\rangle - \langle b_1, b_2\rangle^2 > 0$ (it is $\geqslant 0$ because of the Cauchy Schwarz inequality $|\langle x, y\rangle| \leqslant \|x\|\|y\|$; it cannot be $= 0$, because the Cauchy-Schwarz inequality becomes an equality only when the vectors are parallel, and in our case, b_1, b_2 are linearly independent). Therefore, K has the same sign as $\det(H)$ (**this is always true!** The discussion of this paragraph applies to any surface). Thus, $K \leqslant 0$.

(b) If $K = 0$, then, in view of the above discussion, we must have $\det(H) = 0$, which, in view of our calculation of $\det(H)$, implies that $\langle \dot\gamma(u) \times b(u), \dot b(u)\rangle = 0$, i.e. the mixed product of $\dot\gamma, b, \dot b$ vanishes; this is equivalent to the statement that these three vectors are linearly dependent.

B. Bonus exercise (on geodesics of surfaces of revolution)

12.7 The goal of this exercise is to determine all geodesics of surfaces of revolution. Consider the surface of revolution $\psi : \Omega = [0, 2\pi] \times I \to S \subset \mathbb{R}^3$ given by

$$\psi(\theta, s) = (\, r(s)\cos\theta,\ r(s)\sin\theta,\ z(s)\,).$$

(a) Let $\gamma(t)$ $(t \in J)$ be a C^2 curve traced on the surface S. Show that for all $t \in J$,

$$\langle \ddot\gamma(t), \frac{\partial\psi}{\partial\theta}\rangle = \frac{d}{dt}\big(r^2(t)\,\dot\theta(t)\big),$$

where a dot denotes differentiation with respect to the parameter t.

(b) Show that if $\gamma(t)$ is a geodesic, then the quantities

$$r^2(t)\dot\theta(t) \qquad \text{and} \qquad \big(r(t)\dot\theta(t)\big)^2 + \dot r(t)^2 + \dot z(t)^2$$

are constant (independent of t).

Remark. The first of the above “conservation laws” is related to the fact that translations in θ (i.e. rotations of the surface) are isometric motions, i.e. preserve the geometry of the surface. This can be seen as a special case of the more general *Noether's theorem* for equations arising from a Lagrangian, applied to the case of the equations satisfied by a geodesic. The second conservation law is simply the fact that $\|\dot\gamma\|$ is constant for a geodesic.

(c) Show that the function $t \mapsto 1/r(t)$ is bounded for any geodesic that is not a meridian of the surface of revolution.

Solution. (a) The curve γ can be written as

$$\gamma(t) = \psi(\theta(t), s(t)) = (r(s(t))\cos\theta(t),\ r(s(t))\sin\theta(t),\ z(s(t))).$$

We write $r(t)$ for $r(s(t))$ and $z(t)$ for $z(s(t))$. Note that

$$\dot r = \frac{dr}{ds}\frac{ds}{dt}, \qquad \dot z = \frac{dz}{ds}\frac{ds}{dt}.$$

Then

$$\dot\gamma = \dot r \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + r\dot\theta \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} + \dot z \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Differentiating:

$$\ddot\gamma = \ddot r \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + (2\dot r\dot\theta + r\ddot\theta) \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} - r\dot\theta^2 \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + \ddot z \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

We also have

$$\frac{\partial\psi}{\partial\theta} = r\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}.$$

Thus

$$\left\langle \ddot\gamma, \frac{\partial\psi}{\partial\theta}\right\rangle = r(2\dot r\dot\theta + r\ddot\theta) = \frac{d}{dt}(r^2\dot\theta).$$

(b) If γ is geodesic, then from (a),

$$\frac{d}{dt}(r^2\dot\theta) = 0,$$

so $r^2\dot\theta$ is constant.

We also know that the speed of a geodesic is constant:

$$\|\dot\gamma\|^2 = \dot r^2 + (r\dot\theta)^2 + \dot z^2 = \text{constant}.$$

(c) From (b), there exist constants a, b such that

$$r^2\dot\theta = a, \qquad (r\dot\theta)^2 + \dot r^2 + \dot z^2 = b, \quad b > 0.$$

Thus

$$\frac{a^2}{r^2} = (r\dot\theta)^2 = b - \dot r^2 - \dot z^2 \leqslant b.$$

Two cases:

- If $\dot\theta$ vanishes somewhere, then $a = 0$ and $\dot\theta$ vanishes everywhere. Thus θ is constant and the curve is a meridian.
- Otherwise $a \neq 0$ and $\dot\theta$ never vanishes. Then

  $$\frac{1}{r(t)} \leqslant \frac{\sqrt{b}}{|a|}$$

  so $1/r(t)$ is bounded.